

# Estimate of the three-loop contribution to the QCD static potential from renormalon cancellation

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(Dated: February 1, 2008)

It had been known that the Borel transforms of the twice quark pole mass  $2m_q$  and of the QCD  $q\bar{q}$  static potential  $V(r)$  have leading infrared renormalons at  $b = 1/2$  such that they cancel in the sum. The renormalon residue of  $2m_q$  had been determined with reasonably high precision from the known perturbative coefficients of the ratio  $m_q/\bar{m}_q$ , where  $\bar{m}_q$  is the  $\overline{\text{MS}}$  renormalon-free mass of the quark  $q$ . These values of the residues are used here to estimate the hitherto unknown part of the three-loop coefficient of the static potential  $V(r)$ . Further, the method takes into account the fact that the ultrasoft energy regime contributions must be excluded from the analysis. In the  $b\bar{b}$  quarkonium, aforementioned estimated part of the three-loop term turns out to give a contribution to the binding energy comparable to the nonperturbative contribution.

This is the version v3 as it will appear in J. Phys. G. The changes in comparison to the previous version: the first paragraph is new; the paragraph containing Eqs. (31)-(32) is new, as is Table I; the last paragraph (before Acknowledgments) is new; some new references were added. Numerical results are unchanged.

PACS numbers: 12.38.Bx, 12.38.Cy, 12.38.Aw

The physics of heavy quarkonia is an interesting area of QCD because these systems are expected to be described to a significant degree, but not entirely, by results of perturbative QCD (pQCD). While the perturbation theory cannot explain the binding phenomenon, the values of the binding energies of the ground state and of some excited states of  $b\bar{b}$  and  $t\bar{t}$  are expected to be predicted to a reasonable degree by pQCD, but those of  $c\bar{c}$  less well. Since masses of some of the heavy quarkonia ( $b\bar{b}$  and  $c\bar{c}$ ) states are well measured, their values can be confronted with predictions based on pQCD, where estimates of the nonperturbative corrections should also be included. Such analyses can give us information on the mass of the constituents (e.g., of the heavy quark  $b$ ) and predict masses of other excited states of quarkonium. One of the important quantities used in such analyses is the  $q\bar{q}$  static potential  $V(r)$ . The *perturbative* part of  $V(r)$  is the inverted Fourier transform of the scattering amplitude for the slowly moving  $q\bar{q}$  pair in the static limit  $m_q \rightarrow \infty$ . In general, however,  $V(r)$  is not reducible to a perturbative scattering amplitude that conserves all components of the total momentum of the two incoming and outgoing fermions on their mass-shells. In this work we will estimate the unknown part of the three-loop coefficient of the perturbative static potential, using the known property [1, 2, 3] that the leading infrared renormalon singularities cancel in the sum  $V(r) + 2m_q$ , where  $m_q$  is the pole mass of the quark.

The perturbative expansion of the QCD  $q\bar{q}$  static potential is presently known to two loops [4, 5]. In the three-momentum  $\mathbf{k}$ -space, where  $k = (0, \mathbf{k})$  and  $\mathbf{k} = \mathbf{p}' - \mathbf{p}$  is the three-momentum transfer between the quark and antiquark, the potential is written as [5, 6, 7]

$$V(|\mathbf{k}|) = -\frac{16\pi^2}{3} \frac{1}{|\mathbf{k}|^2} a(\mu) \left\{ 1 + a(\mu) \left[ \frac{1}{4} a_1 + \beta_0 L \right] + a(\mu)^2 \left[ \frac{1}{4^2} a_2 + \left( \frac{1}{2} a_1 \beta_0 + \beta_1 \right) L + \beta_0^2 L^2 \right] \right. \\ \left. + a(\mu)^3 \left[ \frac{1}{4^3} a_3 + b_3 \ln \left( \frac{\mu_f^2}{|\mathbf{k}|^2} \right) + \left( \frac{3}{16} a_2 \beta_0 + \frac{1}{2} a_1 \beta_1 + \beta_2 \right) L + \left( \frac{3}{4} a_1 \beta_0^2 + \frac{5}{2} \beta_0 \beta_1 \right) L^2 + \beta_0^2 L^3 \right] + \mathcal{O}(a^4) \right\}. \quad (1)$$

Here,  $a(\mu) = \alpha_s(\mu, \overline{\text{MS}})/\pi$ ;  $L = \ln(\mu^2/|\mathbf{k}|^2)$  where  $\mu$  is the renormalization scale;  $\beta_j$  ( $j = 0, 1, 2, \dots$ ) are the beta-coefficients appearing in the renormalization group equation (RGE)

$$\frac{da(\mu)}{d \ln \mu^2} = -\beta_0 a^2(\mu) - \beta_1 a^3(\mu) - \beta_2 a^4(\mu) - \dots \quad (2)$$

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We have  $\beta_0 = (11 - 2n_f/3)/4$ ,  $\beta_1 = (102 - 38n_f/3)/16$ . The higher  $\beta_j$  ( $j \geq 2$ ) are renormalization scheme dependent, and are here taken in  $\overline{\text{MS}}$  scheme. Here,  $n_f = n_\ell$  is the number of the light quark flavors active in the soft/potential scale regime  $|k^2|^{1/2} \sim m_q \alpha_s$  ( $n_f$  is four for  $b\text{-}\bar{b}$ , five for  $t\text{-}\bar{t}$ ). The terms involving powers of  $L$  in Eq. (1) are fixed by the renormalization scale independence of  $V$ . The one- and two-loop coefficients  $a_1$  and  $a_2$  were obtained in [8] and [5], respectively

$$a_1 = \frac{1}{9}(93 - 10n_f), \quad (3)$$

$$a_2 = 456.749 - 66.3542n_f + 1.23457n_f^2. \quad (4)$$

At the three-loop level [terms  $\sim a^4$  in  $V(|\mathbf{k}|)$  of Eq. (1)], the coefficient includes the unknown term  $a_3$  and a term with the infrared (IR) cutoff  $\mu_f$  which cuts out the ultrasoft (US) region ( $|k^2|^{1/2} \sim m_q \alpha_s^2 \sim E_{\text{US}}$ ) from the soft/potential (S) region ( $|k^2|^{1/2} \sim m_q \alpha_s \sim E_{\text{S}}$ ):  $E_{\text{S}} < \mu_f < E_{\text{US}}$ . The existence of the IR divergent terms at  $\sim a^4$  in the static Wilson loop was pointed out in Ref. [9]. The IR cutoff term  $b_3 \ln(\mu_f^2/|\mathbf{k}|^2)$  in Eq. (1) has  $b_3 = 9\pi^2/8$ , according to Ref. [10], and is obtained when a Green function involving color-singlet wave function of  $q\text{-}\bar{q}$  is matched in two adjacent effective theories: nonrelativistic QCD (NRQCD) [11] where the hard scales ( $\sim m_q$ ) are integrated out, and the potential NRQCD (pNRQCD) [12] where the soft scales ( $\sim \alpha_s m_q$ ) are integrated out.<sup>1</sup>

The three-dimensional Fourier transformation of expansion (1) gives the perturbative expansion of the static potential in the position-space

$$\begin{aligned} V(r) = & -\frac{4\pi}{3} \frac{1}{r} a(\mu) \left\{ 1 + a(\mu) \left[ \frac{1}{4} a_1 + 2\beta_0 \ell \right] \right. \\ & + a(\mu)^2 \left[ \frac{1}{16} a_2 + (a_1 \beta_0 + 2\beta_1) \ell + \beta_0^2 (4\ell^2 + \pi^2/3) \right] \\ & + a(\mu)^3 \left[ \frac{1}{4^3} a_3 + 2b_3 \ln(\mu_f r e^{\gamma_E}) + \left( \frac{3}{16} \beta_0 a_2 + \frac{1}{2} \beta_1 a_1 + \beta_2 \right) 2\ell \right. \\ & \left. \left. + \left( \frac{3}{4} \beta_0^2 a_1 + \frac{5}{2} \beta_0 \beta_1 \right) (4\ell^2 + \pi^2/3) + \beta_0^3 (8\ell^3 + 2\pi^2 \ell + 16\xi(3)) \right] + \mathcal{O}(a^4) \right\}, \end{aligned} \quad (5)$$

where we use the notation  $\ell = \ln[\mu r \exp(\gamma_E)]$ , with  $\gamma_E$  the Euler constant ( $\gamma_E = 0.5772\dots$ ).

The term  $a_3$  is the only part of the three-loop coefficient that has not been calculated in the literature. It is the value of  $a_3$  that will be estimated by using the aforementioned known property that the leading infrared renormalon ( $b = 1/2$ ) singularities in the sum  $V(r) + 2m_q$  cancel,  $m_q$  being the pole mass of the quark.

The  $b = 1/2$  renormalon of  $m_q$  [14] has a residue that has been determined with a reasonably high precision [15, 16], due to a good convergence behavior of the series for the residue. First the calculation of this residue will be re-done, focusing on the resulting uncertainties. The perturbation expansion for the ratio  $S = m_q/\overline{m}_q - 1$ , where  $\overline{m}_q = \overline{m}_q(\mu = \overline{m}_q)$  is the  $\overline{\text{MS}}$  renormalon free mass, is known to order  $\sim \alpha_s^3$  [17]

$$S \equiv \frac{m_q}{\overline{m}_q} - 1 = \frac{4}{3} a(\mu) \left\{ 1 + a(\mu) [\kappa_1 + \beta_0 L_m] + a(\mu)^2 [\kappa_2 + (2\kappa_1 \beta_0 + \beta_1) L_m + \beta_0^2 L_m^2] + \mathcal{O}(a^3) \right\}, \quad (6)$$

$$(4/3) \kappa_1 = 6.248\beta_0 - 3.739, \quad (7)$$

$$(4/3) \kappa_2 = 23.497\beta_0^2 + 6.248\beta_1 + 1.019\beta_0 - 29.94, \quad (8)$$

and  $L_m = \ln(\mu^2/\overline{m}_q^2)$ . The Borel transform is thus known to order  $\sim b^2$

$$B_S(b; \mu) = \frac{4}{3} \left[ 1 + \frac{r_1(\mu)}{1! \beta_0} b + \frac{r_2}{2! \beta_0^2} b^2 + \mathcal{O}(b^3) \right], \quad (9)$$

where  $r_1 = (\kappa_1 + \beta_0 L_m)$  and  $r_2$  are the NLO and NNLO coefficients in the expansion (6), respectively. On the other hand, this transform can be rewritten as

$$B_S(b; \mu) = \frac{\mu}{\overline{m}_q} \frac{\pi N_m}{(1 - 2b)^{1+\nu}} [1 + \mathcal{O}(1 - 2b)] + (\text{analytic term}), \quad (10)$$

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<sup>1</sup> In Ref. [6], the authors calculated the N<sup>3</sup>LO Hamilton of pNRQCD (using an estimate for  $a_3$  of Ref. [13]) using a method called threshold expansion where all the loop integrations were performed in  $(3 - 2\epsilon)$  dimensions. This generated some additional non-physical terms in the ultrasoft and in the soft/potential regime, but those terms canceled out in the sum from both regimes.

where  $\nu = \beta_1/(2\beta_0^2)$  is the power of the leading ( $b = 1/2$ ) renormalon singularity, the term in the brackets is independent of  $\mu$ , and the last term is a function analytic in the disk  $|b| < 1$  (the other renormalons are at  $b = -1, -2, \dots; +3/2, +2, \dots$ , by Refs. [18, 19]). The  $\mu$ -independent residue parameter  $N_m$  can be obtained by constructing the expansion of the following function in the powers of  $b$ :

$$R_S(b; \mu) \equiv \frac{\overline{m}_q}{\mu} \frac{1}{\pi} (1 - 2b)^{1+\nu} B_S(b; \mu). \quad (11)$$

This function has no pole at  $b = 1/2$ , only a cut at  $b \geq 1/2$  which is a softer singularity. The knowledge of expansion (9) up to  $\sim b^2$  results in the knowledge of expansion of  $R_S$  up to  $\sim b^2$ . In analogy with an observation in Refs. [20], the residue parameter  $N_m$  is

$$N_m = R_S(b=1/2), \quad (12)$$

and can thus be obtained approximately from the mentioned quadratic polynomial (truncated perturbation series - TPS) of  $R_S$  in  $b$ . The convergence of the TPS is good for  $\mu \sim \overline{m}_q$ . There is some (unphysical) variation of the TPS predictions when  $\mu$  is varied. Further, if the Padé resummation  $[1/1]_{R_S}(b)$  is made, the variation with  $\mu$  is different. The results (TPS and Padé) for  $N_m$ , as function of  $\mu/\overline{m}_q$ , are given in Figs. 1(a), (b) for  $n_f = 4, 5$ , respectively. The points of zero  $\mu$ -sensitivity (principle of minimal sensitivity - PMS) give us in the case  $n_f = 4$

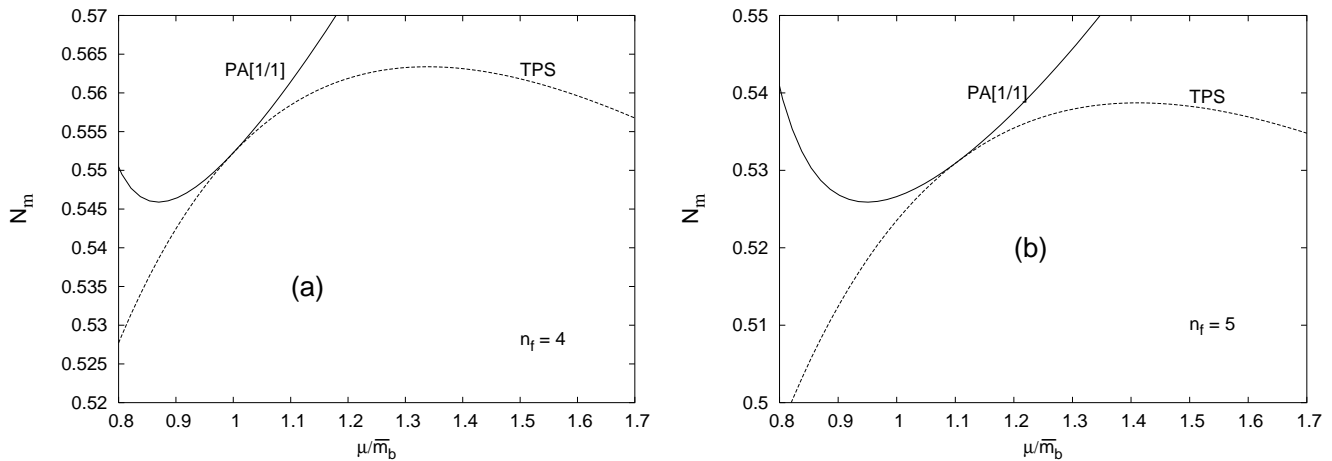


FIG. 1: The residue parameter  $N_m$  determined from (N<sup>2</sup>LO) TPS and Padé  $[1/1]$  of  $R_S(b)$  ( $b=1/2$ ), as function of the renormalization scale parameter  $x = \mu/\overline{m}_q$ , for  $n_f = 4$  (a) and  $n_f = 5$  (b).

predictions  $N_m = 0.563, 0.546$ , for TPS and Padé, respectively, and in the case  $n_f = 5$  predictions  $N_m = 0.539, 0.526$ . As central values we take the arithmetic average between the TPS and Padé values at respective zero  $\mu$ -sensitivities:  $N_m = 0.555(n_f = 4); 0.533(n_f = 5)$ . We could also include in the analysis the Padé predictions at such  $\mu$  where  $b_{\text{pole}}^{\text{Padé}} = -1$ , i.e., the theoretical nearest-to-origin pole of  $R_S(b)$ . However, interestingly, such  $\mu$ 's are close to the Padé-PMS  $\mu$ 's ( $\mu/\overline{m}_q \approx 0.85 - 0.87$  when  $n_f = 4$ ;  $\mu/\overline{m}_q \approx 0.90 - 0.95$  when  $n_f = 5$ ), and the predictions for  $N_m$  are virtually the same. The TPS predictions for  $N_m$  start falling down fast when  $\mu/\overline{m}_q$  decreases below the value one. However,  $\mu/\overline{m}_q < 1$  when the Padé approximant gives the value  $b_{\text{pole}} = -1$  ( $\mu/\overline{m}_q = 0.85, 0.90$  for  $n_f = 4, 5$ , respectively). The absolute value of the uncertainty will be estimated as the deviation of the TPS prediction of  $N_m$  at such low  $\mu$  from the aforementioned central values (0.555, 0.533). The resulting estimates are thus

$$N_m(n_f=4) = 0.555 \pm 0.020, \quad (13)$$

$$N_m(n_f=5) = 0.533 \pm 0.020. \quad (14)$$

On the other hand, the TPS predictions for  $N_m$  at  $\mu/\overline{m}_q = 1$  agree exactly with the values obtained by Pineda [15] ( $N_m = 0.5523, 0.5235$ , for  $n_f = 4, 5$ , respectively), and are well within the limits (13)–(14). Further, in Ref. [16] the values  $N_m \approx 0.557 \pm 0.008, 0.530 \pm 0.008$  for  $n_f = 4, 5$  were obtained using a method involving a conformal mapping, and they are close to the values (13)–(14) here, but the estimated uncertainties here are larger.

A somewhat analogous procedure is now applied to the three-loop expression (5) for the static potential: Requiring the reproduction of the values of  $N_m$  (13)–(14) by the static potential (5) will allow us to estimate the values of the

unknown three-loop coefficient expression  $\tilde{a}_3$  appearing in expansion (5)

$$\tilde{a}_3 \equiv \frac{a_3}{4^3} + \frac{9\pi^2}{4} \ln(\mu_f r e^{\gamma_E}) \quad (15)$$

as a function of the renormalization scale parameter  $\lambda = \mu r$  (it is to be recalled that  $b_3 = 9\pi^2/8$ ). Defining the dimensionless quantity  $F(r) \equiv (-3/(4\pi))rV(r)$ , the TPS (5) gives us the Borel transform of  $F(r)$  up to order  $b^3$

$$B_F(b; \mu) = 1 + \frac{v_1}{1!\beta_0}b + \frac{v_2}{2!\beta_0^2}b^2 + \frac{v_3}{3!\beta_0^3}b^3 + \dots, \quad (16)$$

where  $v_j$  are the coefficients at powers of  $a(\mu)$  in expansion (5) ( $v_1 = a_1/4 + 2\beta_0\ell$ , etc.). This function has the renormalons at  $b = 1/2, 3/2, 5/2$ , etc. [21]. It can be written as

$$B_F(b; \mu r) = \frac{3}{2} \mu r \frac{N_m}{(1-2b)^{1+\nu}} [1 + \mathcal{O}(1-2b)] + (\text{analytic term}), \quad (17)$$

where  $\nu = \beta_1/(2\beta_0^2)$ , the expression in the brackets is  $\mu$ -independent, and the last term is analytic for  $|b| < 3/2$ . The parameter  $N_m$  here is the same as in expression (10) to ensure the leading renormalon cancellation in the Borel transform of the sum  $(V(r) + 2m_q)$ . The function

$$R_F(b; \mu r) \equiv \frac{2}{3} \frac{1}{\mu r} (1-2b)^{1+\nu} B_F(b; \mu r), \quad (18)$$

is then less singular at  $b = 1/2$  (cut instead of pole), and its evaluation at  $b = 1/2$  should give us

$$R_F(b=1/2; \mu r) = N_m. \quad (19)$$

The coefficients  $v_1$  and  $v_2$  appearing in expansions (5) and (16) are explicitly known, but  $v_3$  contains the undetermined three-loop parameter  $\tilde{a}_3$  (15) [ $\Leftrightarrow a_3$ ]. This implies that the coefficients in the expansion of  $R_F(b)$  are known explicitly up to terms  $b^2$ , but the coefficient at  $b^3$  contains  $\tilde{a}_3$ . Thus, any resummation of the N<sup>3</sup>LO TPS of  $R_F(b)$  (i.e., the TPS including the  $b^3$ -term) at  $b = 1/2$ , and requiring the identity (19), with  $N_m$  values (13)–(14), gives us in principle an (approximate) value of the unknown three-loop coefficient  $\tilde{a}_3$ .

From the purely perturbative point of view, the procedure indicates inconsistency because we are matching the residue parameter obtained on the basis of the N<sup>2</sup>LO TPS for  $R_S(b)$  (11) with the residue parameter obtained on the basis of the N<sup>3</sup>LO TPS for  $R_F(b)$  (18). However, in practical terms, there appears to be no inconsistency. The reason lies in the considerably better convergence behavior of the N<sup>2</sup>LO TPS of  $R_S(b)$  than the N<sup>2</sup>LO TPS of  $R_F(b)$  at  $b = 1/2$ . The smallness of the term  $\sim b^2$  in  $R_S(b)$  does not seem to be accidental, because this term is small for any relevant value of  $n_f$  ( $n_f \leq 5$ ). Furthermore, for these reasons, the procedure of estimating  $\tilde{a}_3$  is indeed rather stable, because the N<sup>3</sup>LO term ( $\sim b^3$ ) in  $R_F(b)$  now plays an important role in ensuring numerically the  $b = 1/2$  renormalon cancellation condition (19).

On the other hand, since  $R_F(b)$  is a significantly less singular function than  $B_F(b)$  and at the same time the TPS of  $R_F(b)$  shows only slow convergence (at  $b = 1/2$ ), it is very reasonable to apply some form of resummation beyond the simple TPS evaluation – e.g., Padé resummation. The most stable results for  $\tilde{a}_3$ , when  $\mu$  is varied, are obtained by using the [2/1]( $b$ ) Padé of (the N<sup>3</sup>LO TPS of)  $R_F(b)$ . The resulting values of the parameter  $\tilde{a}_3$ , as function of the renormalization scale parameter  $\lambda = \mu r$  and requiring the central values  $N_m$  of Eqs. (13)–(14), are presented in Figs. 2(a),(b), for the cases  $n_f = 4, 5$ , respectively. When Padé [2/1] resummation is applied to  $R_F(b)$ , the PMS points do exist, with the values  $\tilde{a}_3 = 77.9, 45.2$  at  $\lambda_{\text{PMS}} \approx 0.56, 0.60$ , when  $n_f = 4, 5$ , respectively. If the values of  $N_m$  increase, so does  $\tilde{a}_3$ . The final result is

$$\tilde{a}_3(n_f=4) = 77.9 \pm_{-10.1}^{+9.5}(\delta N_m) \pm 13.2(\delta\mu), \quad (20)$$

$$\tilde{a}_3(n_f=5) = 45.2 \pm_{-8.3}^{+7.7}(\delta N_m) \pm 9.1(\delta\mu). \quad (21)$$

where the first uncertainty is due to the uncertainty  $\pm 0.020$  in the values of  $N_m$  (13)–(14) (Padé [2/1] was evaluated at the unchanged aforementioned values of  $\lambda$ ), and the second uncertainty is due to the renormalization scale dependence. The latter uncertainty was evaluated as plus/minus the deviation of  $\tilde{a}_3$  from the mean value when  $\lambda \equiv \mu r$  is increased by factor two from its PMS value.

The results (20)–(21) for the three-loop parameter  $\tilde{a}_3$  (15) can be extended further, in order to estimate the three-loop coefficient  $a_3$ . This will involve further uncertainties of the S-US factorization scale  $\mu_f$  and of the typical distance  $r$  between the quarks. As emphasized in Ref. [10], the considered  $q\text{--}\bar{q}$  singlet static potential is not the quantity defined

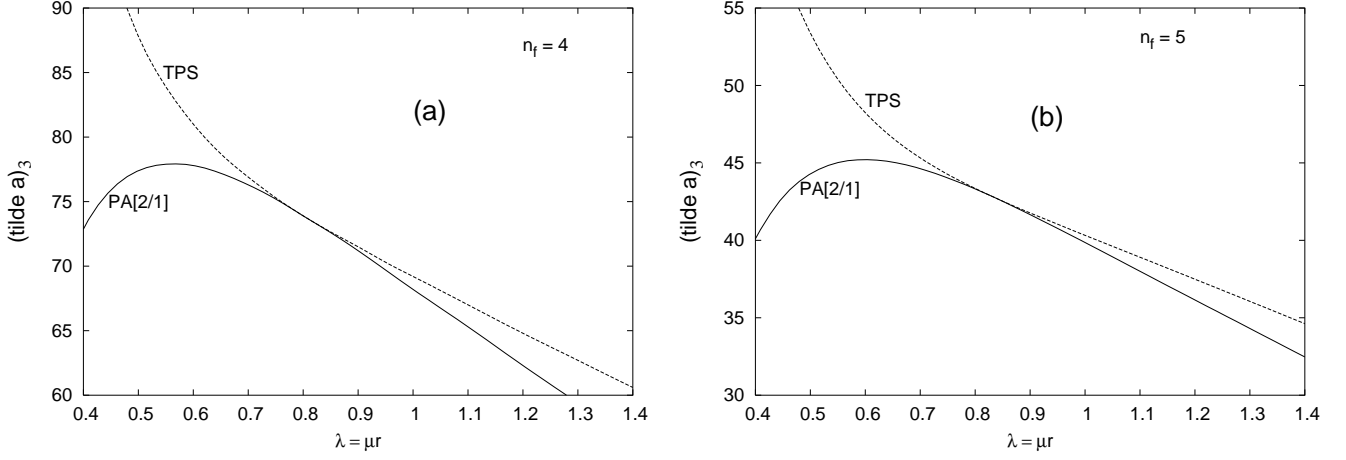


FIG. 2: The three-loop parameter  $\tilde{a}_3$  determined from N<sup>3</sup>LO TPS and Padé [2/1] of  $R_F(b)$  ( $b=1/2$ ) and requiring the central values (13)–(14) for the residue parameter  $N_m$ , as function of the renormalization scale parameter  $\lambda = \mu_r$ , for  $n_f = 4$  (a) and  $n_f = 5$  (b).

via the vacuum expectation value of a static Wilson loop which includes US contributions, but rather the relevant object for the dynamics of the  $q\bar{q}$  pairs with large but finite mass. The kinetic and the binding energies of the  $q\bar{q}$  system are US energies. Therefore, the interactions of the US gluons with the  $q\bar{q}$  system are sensitive to the energies of that system and should be excluded from the static potential [10]. The exclusion of these interactions leads to the term  $\propto \ln(\mu_f r e^{\gamma_E})$  at  $\sim a^4$  in the potential (5). Hence the value of the parameter  $a_3$  will depend on the values of the scales  $\mu_f$  and  $r$ . The dependence of  $V(r)$  on the IR cutoff  $\mu_f$  of the soft scale regime, and the consequent dependence of the values of  $N_m$  (19) and  $a_3$  [cf. Eq. (15)] on  $\mu_f$ , does not imply that US contributions affect them, but rather that the US contributions to those quantities are cut out, zero.

Since the scales  $\mu_f$  and  $r$  appear in logarithms, a natural choice for  $\mu_f$  would be  $\mu_f = (E_S E_{US})^{1/2}$ . The typical S scale is of the order of the three-momentum transfer  $|\mathbf{k}|$  in the  $q\bar{q}$  ground state  $|1\rangle$ :  $E_S \sim m_q \alpha_s$ . The typical US scale is  $E_{US} \sim |E_{q\bar{q}}| \sim m_q \alpha_s^2$ . Therefore, the factorization scale can be estimated as

$$\mu_f \left[ \approx (E_S E_{US})^{1/2} \right] = \kappa m_q \alpha_s (\mu_s)^{3/2}, \quad (22)$$

where  $\kappa \sim 1$  and we take  $\mu_s \approx \mu$  ( $\sim E_S$ ). The typical distance  $r$  in the combination  $\tilde{a}_3$  (15) can be estimated in the following way. First the expectation value of the static potential term proportional to  $\tilde{a}_3$  is calculated in the  $q\bar{q}$  ground state  $|1\rangle$

$$\langle 1 | \frac{1}{r} a^4(\mu) \tilde{a}_3(r) | 1 \rangle = \frac{1}{\tilde{a}_B(\mu)} a^4(\mu) \left[ \frac{a_3}{4^3} + \frac{9\pi^2}{4} \ln(\mu_f \tilde{a}_B(\mu) e/2) \right] \quad (23)$$

$$= \frac{1}{\tilde{a}_B(\mu)} a^4(\mu) \left[ \frac{a_3}{4^3} + \frac{9\pi^2}{4} \ln \left( \frac{3e\kappa}{4\tilde{\kappa}} \alpha_s^{1/2}(\mu_s) \right) \right]. \quad (24)$$

Here,  $\tilde{a}_B(\mu) = 3/(2m_q \tilde{\alpha}_s(\mu))$  is the modified Bohr radius [22] appearing in the  $q\bar{q}$  ground state wavefunction, and we denoted

$$\tilde{\kappa} = \frac{\tilde{\alpha}_s(\mu)}{\alpha_s(\mu)} = 1 + a(\mu) \left[ \frac{1}{4} a_1 + 2\beta_0 \gamma_E \right] + a(\mu)^2 \left[ \frac{1}{16} a_2 + (a_1 \beta_0 + 2\beta_1) \gamma_E + \beta_0^2 (4\gamma_E^2 + \pi^2/3) \right] + \dots \quad (25)$$

In Eq. (24), the expression (22) for  $\mu_f$  was inserted. Eqs. (23)–(24) suggest that we can estimate the value of  $a_3$  from the value of  $\tilde{a}_3$  via the relation

$$\frac{1}{4^3} a_3 \approx \tilde{a}_3 - \frac{9\pi^2}{4} \ln \left( 2.04 \frac{\kappa}{\tilde{\kappa}} \alpha_s^{1/2}(\mu_s) \right). \quad (26)$$

Now we estimate the values of  $\kappa$  and  $\tilde{\kappa}$  for the  $b\bar{b}$  and  $t\bar{t}$  system.

Since  $\mu_s \sim m_q \alpha_s(\mu_s)$ , we have for the  $b\bar{b}$  system ( $m_q \approx 5$  GeV,  $n_f = n_l = 4$ ), the values  $\mu_s \approx 1$ –2 GeV and  $\alpha_s(\mu_s) \approx 0.3$  (as suggested by resummations of the semihadronic  $\tau$ -decay width [23]). The series (25) is strongly

TABLE I: The separate contributions of the perturbative potential  $V(r)$  (5) for two typical radii  $r$  of the  $b\bar{b}$  system ( $n_f=4$ ). The  $a_3$ -contribution is given for the central value (29) of  $a_3$ . The values are given for the renormalization scale  $\mu = 2/r$ , and for  $\mu = 1/r$  in parentheses. Included are also values of the logarithmic confining potential of Ref. [26]. All energies are in MeV; radii are in  $\text{GeV}^{-1}$ . Other details are given in the text.

$r$	$V_{\text{conf}}$	$V(a_3)$	$V(\text{LO})$	$V(\text{NLO})$	$V(\text{N}^2\text{LO})$	$V(\text{N}^3\text{LO})$	$\alpha_s(\mu)$
0.50	+19	-23(-74)	-629(-842)	-319(-328)	-281(-374)	-326(-553)	0.236(0.316)
0.75	+43	-29(-126)	-491(-710)	-292(-350)	-301(-505)	-409(-944)	0.276(0.399)

divergent then  $(1 + 0.37 + 0.40 + \dots)$ , and we can estimate  $\tilde{\kappa}(n_f=4) \approx 1.25 \pm 0.10$  by applying the resummation method of [24].<sup>2</sup> The resummed binding energy in the ground state is  $E_{b\bar{b}} \approx -0.4$  GeV [16, 25], i.e.,  $E_{\text{US}} \approx |E_{b\bar{b}}| \approx 0.9 m_b \alpha_s(\mu_s)^2$ . On the other hand,  $E_S$  is approximately equal to the expectation value of the three-momentum in the ground state

$$E_S \approx \langle 1 | |\mathbf{p}| | 1 \rangle = \frac{8}{3\pi} \frac{1}{\tilde{a}_B(\mu)} = \frac{16}{9\pi} \tilde{\kappa} m_q \alpha_s(\mu) . \quad (27)$$

Therefore, Eq. (22) gives  $\kappa \approx 0.7\tilde{\kappa}^{1/2} (\approx 0.8)$  when  $n_f=4$ .

In the case of the  $t\bar{t}$  system ( $m_q \approx 175$  GeV,  $n_f = n_t = 5$ ), we have  $\mu_s \approx 30$  GeV, and  $\alpha_s(\mu_s) \approx 0.14$ . The series (25) is less divergent  $(1 + 0.15 + 0.07 + \dots)$ , and we estimate with the aforementioned resummation  $\tilde{\kappa}(n_f=5) \approx 1.30 \pm 0.10$ . The resummed binding energy in the ground state is  $E_{t\bar{t}} \approx -3$  GeV [16, 25], i.e.,  $E_{\text{US}} \approx |E_{t\bar{t}}| \approx 0.9 m_t \alpha_s(\mu_s)^2$ . Eqs. (22) and (27) then give  $\kappa \approx 0.7\tilde{\kappa}^{1/2} (\approx 0.8)$  when  $n_f=5$ .

Therefore, the approximate relation (26) can then be rewritten explicitly for the  $b\bar{b}$  and  $t\bar{t}$  system as

$$\frac{1}{4^3} a_3 \approx \tilde{a}_3 - \frac{9\pi^2}{4} \ln \left( 1.4 \frac{1}{\tilde{\kappa}^{1/2}} \alpha_s^{1/2}(\mu_s) \right) , \quad (28)$$

where, as mentioned, we take the central values:  $\alpha_s(\mu_s) \approx 0.3$  and  $\tilde{\kappa} \approx 1.25$  for  $n_f=4$ ;  $\alpha_s(\mu_s) \approx 0.14$  and  $\tilde{\kappa} \approx 1.3$  for  $n_f=5$ . This then gives

$$\frac{1}{4^3} a_3(n_f=4) = 86.3 \pm_{10.1}^{9.5}(\delta N_m) \pm 13.2(\delta\mu) \pm_{8.4}^{15.4}(\delta\mu_f) , \quad (29)$$

$$\frac{1}{4^3} a_3(n_f=5) = 62.5 \pm_{8.3}^{7.7}(\delta N_m) \pm 9.1(\delta\mu) \pm 15.4(\delta\mu_f) , \quad (30)$$

where the last uncertainty is from the uncertainty in the argument of the logarithm in Eq. (28) involving the estimates of the soft and ultrasoft scales. These uncertainties are large, because the central values in the logarithm of Eq. (28) are only suggestive. The value  $\pm 15.4$  for these uncertainties was obtained by allowing the overall factor 2 or 1/2 in the logarithm in Eq. (28). On the other hand, the smaller downward uncertainty  $-8.4$  in the case  $n_f=4$  is due to the problematically small hierarchy between the soft and ultrasoft scales in this case, and the related minimal requirement that the factorization scale  $\mu_f$  should be below the typical soft scale  $\sim |\mathbf{k}|$ . Stated otherwise, the logarithmic term in Eq. (15) should be negative, or equivalently, the IR cutoff term proportional to  $b_3$  in expansion (5) should give a positive contribution to  $V(r)$  (i.e.,  $\mu_f < 0.56/r$ ).

We now investigate how numerically important the obtained  $a_3$ -terms are with respect to other terms in the binding energy of the quarkonium. For example, in the  $b\bar{b}$  ground state  $\Upsilon(1S)$ , the typical value of the interquark distance is  $r \sim 1/(m_b \alpha_s(\mu_s))$ , i.e.,  $r \approx 0.50\text{--}0.75$   $\text{GeV}^{-1}$ . Table I shows the values of the pure  $a_3$ -term of  $V(r)$  for such  $r$ , for the central  $a_3$  value (29), as well as the values of the entire leading order (LO;  $\sim a/r$ ), NLO ( $\sim a^2/r$ ),  $\text{N}^2\text{LO}$  ( $\sim a^3/r$ ), and  $\text{N}^3\text{LO}$  ( $\sim a^4/r$ ) parts of  $V(r)$  of Eq. (5). These values are given for two choices of the renormalization scale:  $\mu = 2/r, 1/r$ . The values of the  $a_3$ -term as given in the Table should be regarded with some caution though, because they strongly depend on the choice of  $\mu$  and because the perturbative series for  $V(r)$  is asymptotically divergent. This can be seen from the values of different orders of  $V(r)$  given in the Table. Borel-related resummation methods for the binding energy  $E_{b\bar{b}}$  [25] give for the  $a_3$ -term contribution  $(-16 \pm 4)$  MeV, i.e.,  $E_{b\bar{b}}(a_3) - E_{b\bar{b}}(0) \approx (-16 \pm 4)$  MeV, when

<sup>2</sup> Note that the series (25) represents the quantity  $(-3/(4\pi))rV(r)$  of Eq. (5) at  $\mu r = 1$ . The principal value (PV) prescription was taken for the Borel integration over the  $b = 1/2$  renormalon singularity.

the values of  $a_3$  are taken from Eq. (29)<sup>3</sup> and  $\mu \sim E_S$ . In addition, we display in Table I the values of the logarithmic confining potential  $V_{\text{conf}}(r) = (C_F \alpha \mathcal{L}/\pi) V_0(\mathcal{L}r)$  of Ref. [26], based on the renormalization-group-improved light-front Hamiltonian formalism of Refs. [27, 28]

$$V_{\text{conf}}(r) = \frac{4}{3\pi} \alpha \mathcal{L} \left[ 2 \ln R - 2 \text{Ci}(R) + \frac{4}{R} \text{Si}(R) - \frac{2}{R^2} (1 - \cos R) + \frac{2}{R} \sin R - 5 + 2\gamma_E \right], \quad (31)$$

where:  $R = \mathcal{L}r$ ;  $\mathcal{L}$  is a cutoff scale parameter of the framework;  $\alpha$  is an effective  $\alpha_s$ . Ci and Si are the usual cosine and sine integrals:  $\text{Ci}(z) = -\int_z^\infty [\cos(t)/t] dt$ ;  $\text{Si}(z) = \int_0^z [\sin(t)/t] dt$ . The fitted values of the parameters in that potential are also taken from Ref. [26]:  $c=1$  [ $c \equiv (4/3)(m_b \alpha/\mathcal{L})$ ],  $m_b = 4.8$  GeV,  $\alpha = 0.4$  ( $\Rightarrow \mathcal{L} = 2.56$  GeV). The nonperturbative contribution to the binding energy  $E_{b\bar{b}}$  can alternatively be estimated as coming from the gluonic condensate [29]

$$E_{b\bar{b}}(us)^{(\text{np})} \approx \bar{m}_b \pi^2 \frac{624}{425} \left( \frac{4}{3} \bar{m}_b \alpha_s(\mu_{us}) \right)^{-4} \langle a(\mu_{us}) G_{\mu\nu} G^{\mu\nu} \rangle \approx (50 \pm 35) \text{ MeV}, \quad (32)$$

where  $\bar{m}_b = 4.2$  GeV was taken, and the ultrasoft energy  $\mu_{us} \sim m_b \alpha_s^2 < 1$  GeV was taken equal to  $\mu \approx 1.5\text{--}2.0$  GeV in order to be able to determine  $\alpha_s(\mu_{us})$  still perturbatively [ $\alpha_s(\mu) \approx 0.30\text{--}0.35$ ]. Thus, the confining energies of Table I and of Eq. (32) consistently give nonperturbative contributions to  $E_{b\bar{b}}$  roughly in the range 20-50 MeV, compared to  $(-16 \pm 4)$  MeV of the  $a_3$ -term. We see that the  $a_3$ -term in  $b\bar{b}$  gives a nonnegligible contribution in comparison to the nonperturbative contributions to the binding energy (and thus to the meson mass). On the other hand, for the lighter charmonium the nonperturbative effects are expected to be more important, while for the much heavier toponium less important than the  $a_3$ -term contributions.

The contributions to  $E_{b\bar{b}}$  from the “hard” ( $\sim m_b$ ) and “soft” ( $\sim m_b \alpha_s$ ) energy regimes give about  $-300$  MeV, and those from the ultrasoft regime ( $\sim m_b \alpha_s^2$ ) about  $-150$  MeV [25]. By the virial theorem, this implies for the kinetic energy an estimate of 150-250 MeV.

We now compare the results (20)–(21) and (29)–(30) with those in the literature. The authors of Ref. [13] have estimated the three-loop coefficient of the static potential  $V(|\mathbf{k}|)$  with a Padé-method based on the known one- and two-loop coefficients. However, they did not have the IR cutoff term in the three-loop coefficient. Therefore, there is no direct correspondence with the results presented here, but in  $V(r)$  their coefficient  $c_0$  formally may correspond to  $\tilde{a}_3$  of Eq. (15), or equivalently, to  $a_3/4^3$  if  $\mu_f = \exp(-\gamma_E)/r$ . Their estimated values were  $c_0 = 97.5, 60.1$  for  $n_f = 4, 5$ , which are higher than the values (20)–(21). The estimates of T. Lee [16], also without the effects of the IR cutoff, and based on a Borel transform method [30], give  $a_3/4^3 = 59 \pm 81$  ( $n_f = 4$ );  $34 \pm 63$  ( $n_f = 5$ ). Further, Pineda [15] estimated the three-loop coefficient of  $V(r)$  from renormalon-dominated large order behavior of the coefficients for  $\mu = 1/r$  but without consideration of the IR cutoff effects; if his results are to be interpreted with the value of the factorization scale  $\mu_f = 1/r$  ( $\mu_f \equiv \nu_{us}$  in the notation of [15]), as suggested by him, then this would give, in the notation (15) of the present paper, the values  $a_3/4^3 = 59.6, 24.3$  for  $n_f = 4, 5$ . On the other hand, in terms of  $\tilde{a}_3$  of Eq. (15), his results would imply  $\tilde{a}_3 = 72.4, 37.1$  for  $n_f = 4, 5$ , which are not very far from the values (20)–(21). Pineda, in the framework of his method, did account for the leading IR renormalon cancellation in  $(V(r) + 2m_q)$ .

The main results and conclusions of the present work are the following: The unknown part  $a_3$  of the three-loop coefficient of the static potential (5) was estimated by using the known property that the leading infrared renormalon singularities cancel in the sum  $V(r) + 2m_q$ . Further, the presented method takes into account the fact that the contributions of the ultrasoft energy regime should be excluded from the analysis. The obtained estimated values of  $a_3$  for the  $b\bar{b}$  and  $t\bar{t}$  quarkonia are given in Eqs. (29) and (30), respectively. In the  $b\bar{b}$  system, the obtained value of the three-loop parameter  $a_3$  leads to a decrease of the binding energy by about 10-20 MeV. This is smaller than, but still comparable to, the increase of the binding energy by about 20-50 MeV by nonperturbative effects. The similar value (30) of  $a_3$  for the much heavier  $t\bar{t}$  system is expected to influence the binding energies to a similar degree as in  $b\bar{b}$ , but the nonperturbative effects are expected to be much weaker. For the lighter  $c\bar{c}$  system the nonperturbative effects are expected to dominate over the three-loop effects.

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<sup>3</sup> The uncertainties were added in quadrature:  $a_3/4^3 \approx 86 \pm 23$ .

### Acknowledgments

The author thanks to A.A. Penin for several clarifications, and to C. Contreras and P. Gaete for useful discussions. This work was supported by FONDECYT (Chile) grant No. 1010094.

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